Simple approximate solutions of the radiative transfer equation for a cloudy atmosphere

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ABSTRACT

The paper is devoted to the derivation of approximate analytical equations for the top-of-atmosphere reflection function of a cloudy atmosphere. These equations are based on the analytical solution of the radiative transfer equation valid for optically thick clouds. In particular, we consider the radiative transfer both in the gaseous absorption bands and also in the regions, where gaseous absorption can be neglected. The results obtained are of importance for a number of cloud optics problems including cloud optical and microphysical properties determination from spaceborne optical instruments.

Keywords: radiative transfer, clouds, remote sensing

1. INTRODUCTION

Modern techniques for the retrieval of cloud parameters from satellite measurements are mostly based on the look-up-table (LUT) approach (Nakajima and King, 1990). However, LUTs can be substituted in number of cases by the analytical solutions of the radiative transfer equation valid for optically thick clouds (King et al., 1997; Kokhanovsky et al., 2003). This simplifies and speeds up the retrieval process considerably (Kokhanovsky et al., 2003; Rozanov and Kokhanovsky, 2004). The aim of this work is to introduce a number of analytical equations which can be used to model the top-of-atmosphere satellite signal over an extended cloud field. The work is based on analytical solutions of the integro-differential radiative transfer equation (RTE) derived by Germogenova (1961, 1963) for the case of arbitrary absorbing optically thick light scattering media. We simplify solutions taking into account that light absorption by droplets and crystals in clouds is weak. Also we consider the radiative transfer in gaseous absorption bands.

2. ASYMPTOTIC THEORY

The asymptotic solutions of the radiative transfer equation valid for optically thick media were derived by Germogenova (1961). They can be used for arbitrary local scattering laws and levels of absorption in the medium. The only limitation is that the optical thickness τ must be a large number. King (1987) found that the asymptotic theory is accurate to within 1% for $\tau \geq \tau^*$, where $\tau^* = 1.45\sigma$ with $\sigma = (1-g)^{-1}$. Here g is the asymmetry parameter of the local single scattering law defined as $g = \frac{1}{2}\int_0^\pi p(\theta)\cos\theta\sin\theta d\theta$, where $p(\theta)$ is the probability of light scattering in a given direction or the phase function. It means that $\tau^* = 145$ for biological media having *g=0.99*. Fortunately, the value of g is close to 0.85 for most of water clouds in visible (Kokhanovsky, 2004a). This means that $\tau^* = 10$ and asymptotic theory is applicable to most of cases of extended cloudiness. The situation is even better for crystalline clouds. Then the value of g is close to 0.75 and $\tau^* = 6.0$.

The asymptotic solution for the cloud reflection function can be written in the following simple form (Germogenova, 1961, 1963; Sobolev, 1984; Nakajima and King, 1992):

$$R(\mu,\mu_0,\varphi,\tau) = R_\infty(\mu,\mu_0,\varphi) - \frac{mle^{-2k\tau}}{1-l^2e^{-2k\tau}}K(\mu)K(\mu_0)\,. \quad (1)$$

Here $R(\mu,\mu_0,\varphi,\tau)$ is the reflection function for a plane-parallel cloud layer. The values μ and μ_0 are cosines of the observation ϑ and incidence ϑ_0 angles, respectively, and φ is the relative azimuth. The function $R(\mu,\mu_0,\varphi,\tau)$ is defined as the ratio of the intensity I_r of reflected light for a turbid layer illuminated in the direction specified by the incidence angle ϑ_0 to the value of I_r for the absolutely white Lambertian screen. Clearly, we have for the absolutely white Lambertian surfaces: $R \equiv 1$ independently of the observation angle ϑ and the relative azimuth φ.

There is a major problem associated with Eq. (1). It requires quite complex calculations of escape functions $K(\mu)$, reflection functions for a semi-infinite layer $R_\infty(\mu,\mu_0,\varphi)$ and parameters *k, l, m* using integral equations (Minin, 1988). Also matrix equations can be used for this purpose(Nakajima and King , 1992).

The problem is much more simpler in visible, where the absorption of light by clouds is negligible in most of cases and Eq. (1) transforms to the following result (van de Hulst, 1980):

$$R(\mu,\mu_0,\varphi,\tau) = R_\infty^0(\mu,\mu_0,\varphi) - t(\tau)K_0(\mu)K_0(\mu_0). \tag{2}$$

Here $R_\infty^0(\mu,\mu_0,\varphi)$ is the reflection function of a nonabsorbing semi-infinite light scattering layer and $K_0(\mu)$ is the escape function for a nonabsorbing medium, and

$$t(\tau) = \frac{1}{\alpha + 0.75\tau(1-g)} \tag{3}$$

is the diffuse transmittance of a cloud layer under the diffuse illumination. The parameter $\alpha \approx 1.07$ does not vary considerably for media having different microstructures (King, 1987). Also it follows within the accuracy 2% (Kokhanovsky, 2004b):

$$K_0(\mu) = \frac{3}{7}[1+2\mu] \tag{4}$$

at $\mu \geq 0.2$. Effectively, Eqs. (2)-(3) reduce the problem of calculation of the reflection function of a finite cloud to that of a semi-infinite cloud. This is of a great importance because the function $R_\infty^0(\mu,\mu_0,\varphi)$ depends only on the phase function and this dependence is rather weak (Kokhanovsky, 2004a). Therefore, corresponding LUTs in the cloud retrieval algorithms can be substantially reduced (King et al., 1997). Also one can use parameterizations of the function $R_\infty^0(\mu,\mu_0,\varphi)$, which enhances the speed of retrieval even further (Kokhanovsky et al., 2003). In particular, the following parameterization of this function can be used(Kokhanovsky, 2004c):

$$R_\infty^0(\mu,\mu_0,\varphi) = \frac{A + B(\mu+\mu_0) + C\mu\mu_0 + F(\theta)}{4(\mu+\mu_0)}. \tag{5}$$

The parameters *A, B, C* and the function $F(\theta)$ differ for different particulate media and can be found comparing calculations using Eq. (5) with numerical solutions of the RTE (e.g., various least square minimization techniques can be used for this purpose). Note that $\theta = acos(-\mu\mu_0 + ss_0\cos\varphi)$ is the scattering angle ($s_0 = \sin\vartheta_0$, $s = \sin\vartheta$). Kokhanovsky (2004b) has shown that it follows for water clouds: *A=3.944, B=-2.5, C=10.664,* and $F(\theta) = p(\theta) - \bar{p}(\theta)$, where $p(\theta)$ is the phase function. The bar means averaging with respect to the azimuth. Following the same procedure as described by Kokhanovsky (2004b) we have found for crystalline clouds: *A=1.247, B=1.186, C=5.157,* and $F(\theta) \equiv p(\theta)$. Note that for most of applications one can neglect the small contribution due the last term in the nominator of Eq. (5).

It is of importance to have a simple formulation similar to that shown above for water clouds in the near –infrared (e.g., for wavelengths less than 2.25 μm, where many modern spectrometers and radiometers onboard various satellite platforms operate). This could be done either using parameterizations of functions and parameters given in Eq. (1)

against the single scattering albedo ω_0 (King and Harshvardhan, 1986) or applying so-called exponential approximation (Zege et al., 1991). The next section is devoted to the derivation of the analytical expression for the reflection function of cloud fields in the modified exponential approximation.

3. THE EXPONENTIAL APPROXIMATION

The idea of the exponential approximation is quite simple. The main parameters and functions in Eq. (1) have following analytical forms for a small probability of photon absorption (PPA) $\beta = 1 - \omega_0$ in the single photon-particle interaction event (van de Hulst, 1980):

$$R_\infty(\mu,\mu_0,\varphi) = R_\infty^0(\mu,\mu_0,\varphi) - 4\sqrt{\frac{\beta}{3(1\text{-g})}}K(\mu)K(\mu_0), \quad (6)$$

$$K(\mu) = K_0(\mu)\left(1 - 2\alpha\sqrt{\frac{\beta}{3(1\text{-g})}}\right), \quad (7)$$

$$k = \sqrt{3(1-g)\beta}, \quad l = 1 - 4\alpha\sqrt{\frac{\beta}{3(1\text{-g})}}, \quad m = 8\sqrt{\frac{\beta}{3(1\text{-g})}}. \quad (8)$$

These approximations are valid only for values of $\beta = \sigma_{abs}/\sigma_{ext} \le 0.0001$. Here σ_{abs} and σ_{ext} are absorption and extinction coefficients, respectively. For water clouds in the near-infrared the PPA β can reach *0.1* and even larger values (Kokhanovsky, 2004a). So we need to consider next terms in the expansions (6)-(8) (Minin, 1991). However, corresponding expressions appear to be extremely complicated for most of practical applications. Therefore, Zege et al. (1991) proposed following exponential forms for the reflection function $R_\infty(\mu,\mu_0,\varphi)$ and the combination $f(\mu,\mu_0) = mK(\mu)K(\mu_0)$ in Eq. (1):

$$R_\infty(\mu,\mu_0,\varphi) = R_\infty^0 \exp(-yu(\mu,\mu_0,\varphi)), \quad (9)$$

$$f(\mu,\mu_0) = (1 - \exp(-2y))K_0(\mu)K_0(\mu_0), \quad (10)$$

where

$$y = 4\sqrt{\frac{\beta}{3(1-g)}}, \quad (11)$$

$$u(\mu,\mu_0,\varphi) = \frac{K_0(\mu)K_0(\mu_0)}{R_\infty^0(\mu,\mu_0,\varphi)}. \quad (12)$$

Similar exponential expression can be used for the parameter *l*. Namely, we have:

$$l = \exp(-\alpha y). \quad (13)$$

It should be stressed that these expressions transform into the exact asymptotic results given by Eqs. (6)-(8) as $\beta \to 0$. However, they allow us to extend the applicability of Eqs. (6)-(8) to larger values of β. Note that empirical exponential forms appear here not by chance but have deep roots related to the light diffusion theory (Rozenberg, 1961, Kokhanovsky, 2004b). The substitution of Eqs. (9), (10), (13) into Eq. (1) gives:

$$R(\mu,\mu_0,\varphi,\tau) = R_\infty^0 \exp(-yu(\mu,\mu_0,\varphi)) - te^{-x-y}K_0(\mu)K_0(\mu_0), \quad (14)$$

where we introduced a new parameter $x = k\tau$. The global transmittance t is given by:

$$t = \frac{\sinh y}{\sinh(\alpha y + x)}. \quad (15)$$

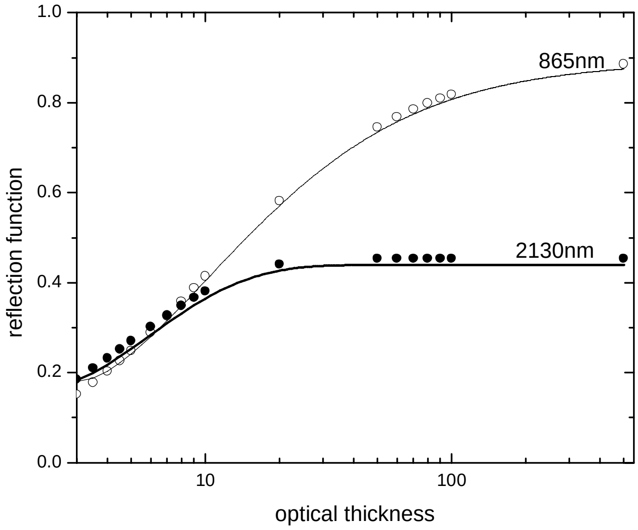

Fig.1 Dependence of the reflection on the optical thickness for wavelengths 865nm and 2130nm at the nadir observation and the solar zenith angle 60 degrees. The effective radius of droplets is 6 μm and the coefficient of variance of the gamma particle size distribution is $1/\sqrt{7}$. Lines give the results according to the MEA. Points show results of exact calculations using the vector radiative transfer code SCIAPOL. The following values of the refractive index *m* of water droplets have been used: *m=1.324*($\lambda = 864nm$) and *m=1.29-0.0004i*($\lambda = 2130nm$).

Eq. (14) transforms into Eq. (2) (and also Eq. (15) transforms into Eq. (3)) as $\beta = 0$. However, Eq. (14) unlike Eq. (2) allows to consider absorbing media as well. It is important that no new angular functions arise in Eq. (14) as compared to Eq. (2). This is in contrast with Eq. (1), where parameters and functions have an implicit and complex dependence on the PPA β. Eq. (14) can be used for the rapid estimations of light reflection from cloudy media and also for speeding up cloud retrieval algorithms (Kokhanovsky et al., 2003). The range of applicability of the exponential approximation (14) can be extended using correction terms derived from the numerical solution of the radiative transfer equation. In particular, we find that the accuracy of Eq. (14) for cloudy media can be increased using following substitutions: $u \to u(1-0.05y)$, $t \to t-\Delta$, where

$$\Delta = \frac{a + b\mu\mu_0 + c\mu^2\mu_0^2}{\tau^3}\exp(x) \qquad (16)$$

and *a=4.86, b=-13.08, c=12.76.* Therefore, the final equation can be written as

$$R(\mu,\mu_0,\varphi,\tau) = R_\infty^0 \exp\left(-y(1-0.05y)u\left(\mu,\mu_0,\varphi\right)\right) - (t-\Delta)e^{-x-y}K_0\left(\mu\right)K_0\left(\mu_0\right). \qquad (17)$$

Eq. (17) is called the modified exponential approximation (MEA). We show the accuracy of the MEA given by Eq. (17) with account for Eqs. (4), (12), (16) in Figs.1, 2 for the nadir observation conditions, the solar zenith angle 60° and wavelengths 865nm and 2130nm. These wavelengths are often used in cloud retrieval techniques. Note that the single scattering albedo is equal to 1.0 and 0.9872 at these wavelengths, respectively. The asymmetry parameter is 0.8435 for the smaller wavelength. It is 0.8054 for the wavelength 2130nm. Exact data shown in Fig.1 are obtained using the vector radiative transfer code SCIAPOL developed by us. The SCIAPOL code is based on the discrete ordinate approach and thoroughly tested against tabular results presented by Siewert(2000). It follows that the accuracy of the approximation is better than 6% for the cloud optical thickness $\tau \geq 4$ in the case considered. Calculations performed for other angles show that the accuracy only weakly depends on the geometry providing that grazing observation and illumination conditions are excluded (Kokhanovsky and Rozanov, 2003). It means that the top-of-atmosphere reflectance over cloudy scenes can be accurately modeled in the framework of the MEA (even as

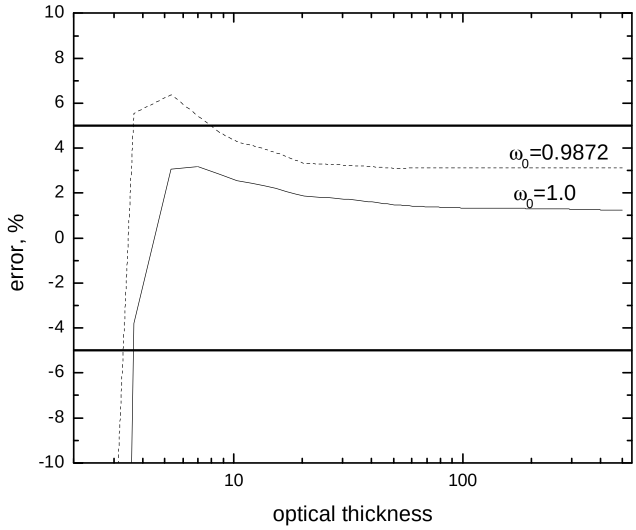

Fig.2. The errors of the MEA found from data given in Fig. 1. The solid line with the higher value of the single scattering albedo corresponds to the wavelength 865nm. The broken line corresponds to results obtained for the wavelength 2130nm.

compared to the vector radiative transfer model). One can see from Fig.2 that the accuracy of the MEA could be increased if the exact result for the reflection function of a semi-infinite layer is used in calculations. Note that we used in Eq. (14) the following simple formula instead of Eq. (5) valid for the nadir observation conditions only (Kokhanovsky, 2002) :

$$R_\infty^0\left(\mu,\mu_0,\varphi\right)=\frac{0.37+1.94\mu_0}{1+\mu_0}\ . \tag{18}$$

The results obtained with this equation are close to those derived from more complex Eq. (5) at $\mu=1$, although Eq. (18) is a lot simpler. Its accuracy can be further increased by adding the function $F=0.25p(1-\arccos(\mu_0))$ to the numerator.

3. THE RADIATIVE TRANSFER IN GASEOUS ABSORPTION BANDS

The exponential approximation presented above can be easily extended to account for the gaseous absorption. Then one should use the following substitutions in equations given above: $\tau\rightarrow\tau+\tau_g$, $\beta\rightarrow(\sigma_{abs}+\sigma_{abs,g})/\left(\sigma_{ext}+\sigma_{abs,g}\right)$, where the subscript "g" relates the correspondent value to the gaseous absorption process. The phase function does not need to be modified because we ignore molecular scattering. This could be easily accounted for if necessary. However, we account for the additional light absorption in the atmosphere above a cloud. Therefore, it follows for the cloud reflection function $\overline{R}$ in the gaseous absorption band : $\overline{R}=T_1RT_2$, where we omitted arguments for the simplicity. The value of R is given by Eq. (17) and $T_j=\exp(-m_j\tau_{abs})$, *j=1,2*, where $m_1=1/\mu_0$, $m_2=1/\mu$, and

$$\tau_{abs}=\sum_{i=1}^{N}\int_{z_1}^{z_2}C_{abs,i}\left(z\right)\varsigma_i\left(z\right)dz\ , \tag{19}$$

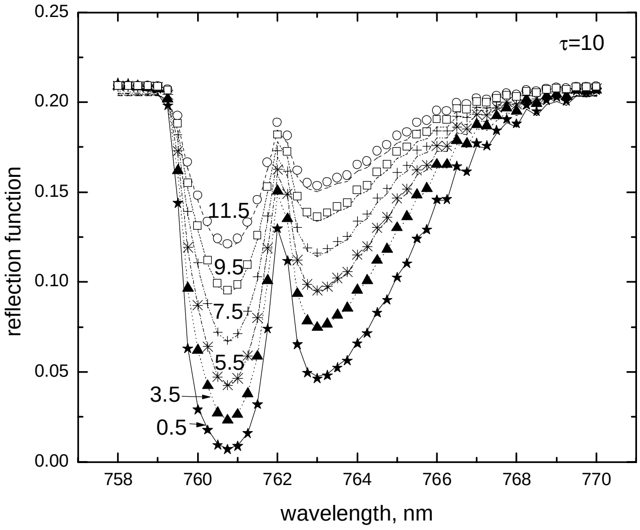

Fig.3 Dependence of the cloud reflection function on the wavelength in the oxygen A-band for cloud top heights equal to 0.5, 3.5, 5.5, 7.5, 9.5, 11.5 km at the cloud optical thickness $\tau = 10$. The cloud geometrical thickness is equal to 250m. The droplet size distribution and the illumination/viewing conditions coincide with that used in calculations presented in Fig.1. The atmospheric model used is identical to that described by Kokhanovsky and Rozanov(2004). Symbols give exact results obtained with SCIATRAN (Rozanov et al., 2005). Lines are plotted using Eq. (20).

where $C_{abs,i}$ is the ith gas absorption cross section, N is the total number of gases present and $\varsigma_i(z)$ is the concentration of the ith gas at a given height. The integration extends from the upper cloud boundary position z_1 to the height of the optical instrument z_2. The accuracy of the MEA for the gaseous absorption band can be increased if the single scattering contribution in the signal from the atmospheric layer above the cloud R_s is also taken into account. Then it follows:

$$\bar{R} = T_1 R T_2 + R_s . \qquad (20)$$

The expression for R_s is presented elsewhere (Kokhanovsky and Rozanov, 2004). We checked the accuracy of Eq. (20) with account for Eqs. (4), (5), (12), (16), (17) by performing exact calculations using the radiative transfer code SCIATRAN (Rozanov et al., 2005) for the oxygen absorption A-band located at the wavelengths 758-768nm. The atmospheric model used in calculations coincides with that described by Kokhanovsky and Rozanov (2004). The values of $\bar{R}$ are averaged with respect to the Gaussian instrument response function with the half-width 0.225nm. The absorption by the oxygen was accounted for by using the HITRAN 2000 (Rothman et al., 2003) database in conjunction with the correlated k-distribution approximation (Kokhanovsky and Rozanov, 2004). To increase the accuracy of the model, we accounted for light scattering and absorption below the cloud layer using the approximate technique developed by Kokhanovsky and Rozanov (2004). Results are given in Figs. 3-6.

It follows from the analysis of the data presented that the accuracy of approximate calculations is better than 5% in most of cases. The errors increase for low clouds having larger values of τ due to the simplicity of our model, which accounts for the cloud – upper atmospheric layer interaction in a first coarse approximation only (Kokhanovsky and Rozanov, 2004). This interaction becomes more important for lower thick clouds (see Fig.6).

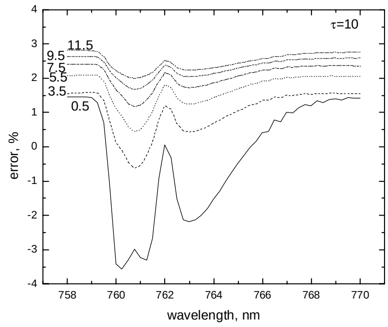

Fig. 4. The errors of Eq. (20) derived using data shown in Fig.3 for various cloud top height positions and $\tau = 10$.

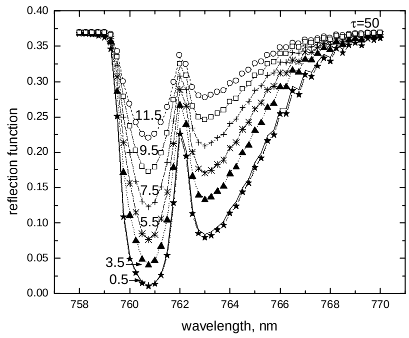

Fig.5. The same as in Fig.3 except at $\tau = 50$.

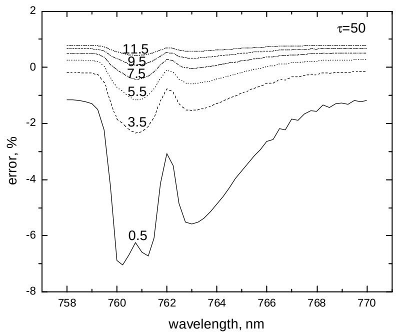

Fig. 6. The errors of Eq. (20) calculated using data shown in Fig.5 for various cloud top height positions and $\tau = 50$.

4. CONCLUSION

We present here the modified exponential approximation for the reflection function of a cloudy field. This approximation has the accuracy better than 5-10% both inside and outside gaseous absorption bands for cloudy media with the optical thickness $\tau \geq 5$ in the visible and near-infrared spectral regions for most of viewing and illumination conditions. Only the case of black underlying surface is considered. However, results can be easily extended for the case of arbitrary underlying Lambertian surfaces (Kokhanovsky and Rozanov, 2004). The exponential-type approximations can be developed for other cloud characteristics including cloud transmission functions and polarization characteristics (Kokhanovsky, 2003). Similar approach could be used for studies of light fields inside

clouds. The SCITRAN 2.0 software package used in this work (Rozanov et al., 2005) is freely available for non-commercial use at the website www.iup.physik.uni-bremen.de/sciatran. This package also includes the possibility to calculate the cloud reflection function in the modified exponential approximation presented here.

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